Eigenstructure Approach for Array Processing with Unknown Intensity Coefficients

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Abstract

The eigenstructure approach for array processing is examined for the general case in which it is required to estimate several parameters related to the directional patterns of the sources as well as parameters related to the location of the sources. The assumption of the MUSIC algorithm that any given source is observed by all the sensors with the same intensity is removed, and hence the proposed technique is useful for localizing emitters in the near field of the array and for using sensors with unknown radiation pattern. The resulting method is illustrated by a simple example, which is also used to show that the standard MUSIC algorithm does not work when the assumption of equal intensities is violated.

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I. Introduction

In this paper we are concerned with the problem of localizing several radiating sources by observation of their signals at spatially separated sensors. This is a problem of considerable importance, occurring in a variety of fields ranging from radar, sonar, and oceanography to seismology and radio-astronomy. In recent years there has been a growing interest in eigenstructure based methods, perhaps due to the introduction of the MUSIC method by Schmidt [1], which is a technique that can be applied to general array configurations and which is relatively simple and efficient. A comprehensive discussion of the MUSIC method may be found in [1], while [3] contains a literature survey of most of the recently published results.

An assumption common to all previously published contributions in this area is that any given source is observed by all the sensors with the same intensity. This assumption is reasonable only if the sources are in the far-field of the array and the sensors have identical radiation patterns. In this paper we illustrate a potential problem with the basic MUSIC method when this assumption is violated. To remedy this problem we remove the assumption of equal intensity and thus extend the applicability of the MUSIC technique, or any other eigenstructure approach, to the case of near field sources and/or sensors with unknown radiation patterns. The paper is organized as follows. The problem formulation and proposed solution are described in Section II. In Section III we illustrate through examples that the MUSIC method breaks down when the signals are observed with unequal intensities, while the proposed technique performs well. However, since in our approach there are more degrees of freedom, spurious estimates may be generated. We indicate how post processing can eliminate the phantom results. Section IV contains some conclusions.

II. Problem Formulation and Solution

Consider N radiating sources with an arbitrary radiation pattern observed by an array of M sensors. The signal at the output of the m-th sensor can be described by

$$x_m(t) = \sum_{n=1}^{N} \alpha_{mn} s_n(t-\tau_{mn}) + v_m(t) \; ; \quad m = 1,2,\cdots,M$$

$$-T/2 \le t \le T/2 \qquad (1)$$

where $\left\{s_n(t)\right\}_{n=1}^{N}$ are the radiated signals, $\left\{v_m(t)\right\}_{m=1}^{M}$ are additive noise processes, and T is the observation interval. The intensities α_{mn} and the delays τ_{mn} are parameters related to the directional patterns and relative location of the n-th source and the m-th sensor.

A convenient separation of the parameters to be estimated is obtained by using Fourier coefficients defined by

$$x_m(\omega_\ell) = \frac{1}{\sqrt{T}} \int_{-T/2}^{T/2} x_m(t) e^{-j\omega_\ell t} dt,$$

where $\omega_\ell = \frac{2\pi}{T}(\ell_1+\ell)$, $\ell = 1,2,\cdots,L$; and ℓ_1 is a constant. In principle the number of coefficients required to capture all the signal information is infinite. However, if we consider signals with energy concentrated in a finite spectral band, we can use only $L < \infty$ coefficients.

Taking the Fourier coefficients of (1) we obtain

$$X_m(\omega_\ell) = \sum_{n=1}^{N} \alpha_{mn} e^{-j\omega_\ell \tau_{mn}} S_n(\omega_\ell) + V_m(\omega_\ell) \; ; \quad \ell = 1,2,\cdots,L \; ; \qquad (2)$$

where $s_n(\omega_\ell)$ and $V_m(\omega_\ell)$ are the Fourier coefficients of $s_n(t)$ and $v_m(t)$ respectively. Equation (2) may be expressed using vector notation as follows:

$$\underline{X}(\omega_\ell) = A(\omega_\ell)\underline{S}(\omega_\ell) + \underline{V}(\omega_\ell) \quad ; \quad \ell = 1,2,\cdots,L \; ; \tag{3}$$

where

$$\underline{X}(\omega_\ell) = \left[X_1(\omega_\ell),\ X_2(\omega_\ell),\cdots,X_M(\omega_\ell)\right]^T$$

$$\underline{S}(\omega_\ell) = \left[S_1(\omega_\ell),\ S_2(\omega_\ell),\cdots,S_N(\omega_\ell)\right]^T$$

$$\underline{V}(\omega_\ell) = \left[V_1(\omega_\ell),\ V_2(\omega_\ell),\cdots,V_M(\omega_\ell)\right]^T$$

$$A(\omega_\ell) = \left[\underline{a}_\ell(\underline{\theta}_1),\ \underline{a}_\ell(\underline{\theta}_2),\cdots,\underline{a}_\ell(\underline{\theta}_N)\right]$$

$$\underline{a}_\ell(\underline{\theta}_n) = \left[\alpha_{1n}e^{-j\omega_\ell\tau_{1n}},\ \alpha_{2n}e^{-j\omega_\ell\tau_{2n}},\cdots,\alpha_{Mn}e^{-j\omega_\ell\tau_{Mn}}\right]^T .$$

We use $\underline{\theta}_n$ to represent all the parameters of interest associated with the n-th signal, namely $\left\{\alpha_{mn}\right\}_{m=1}^{M}$ and $\left\{\tau_{mn}\right\}_{m=1}^{M}$. Our main goal is to estimate the set $\left\{\underline{\theta}_n\right\}_{n=1}^{N}$. Note that if the spectrum of the signals is concentrated around ω_1, with a bandwidth that is small compared to $2\pi/T$, then (3) reduces to a single relation between the observation vector $\underline{X}(\omega_1)$ and the parameters, i.e. $L = 1$. In this case, it is customary to use many short observation intervals or simply time samples, and the model becomes:

$$\underline{X}(j) = A\underline{S}(j) + \underline{V}(j) \quad ; \quad j = 1,2,\cdots,J \; , \tag{4}$$

where the dependence on the single frequency ω_1 is suppressed, and j denotes the index of the different samples. Note that the main difference between the narrowband case and the wideband case is that A is the same in all the J equations specified by (4) while $A(\omega_\ell)$ is different in each of the L equations given by (3). However, the estimation procedure discussed here is equally applicable in both cases. In this communication we concentrate on the narrowband case. The modification for the wideband case, using for example [2] or [4], is straightforward.

The following assumptions are made:

(a) The signals and noises are stationary over the observation interval.

(b) The number of sources is known and is smaller than the number of sensors.

(c) The columns of A are linearly independent.

(d) The signals are not perfectly correlated.

(e) The noise covariance matrix is known except for a multiplicative constant σ^2.

The correlation matrices of the signal, noise and observation vectors are given respectively by

$$R_s = E\left\{\underline{S}\underline{S}^H\right\}$$

$$\sigma^2\Sigma_0 = E\left\{\underline{N}\underline{N}^H\right\}$$

$$R_x = E\left\{\underline{X}\underline{X}^H\right\} = AR_sA^H + \sigma^2\Sigma_0 \tag{5}$$

where $(\cdot)^H$ represents the Hermitian transpose operation. The following theorem forms the basis for the eigenstructure approach.

Theorem: Let λ_i and $\underline{u}_i$, $i = 1,2,\cdots,M$ be the eigenvalues and corresponding eigenvectors of the matrix pencil (R_x,Σ_0), (i.e. the solutions of $R_x\underline{u} = \lambda\Sigma_0\underline{u}$), where the λ_is are listed in descending order. Then,

(1) $\lambda_{N+1} = \lambda_{N+2} = \cdots = \lambda_M = \sigma^2$.

(2) Each of the columns of A is orthogonal to the matrix

$U = [\underline{u}_{N+1},\underline{u}_{N+2},\cdots,\underline{u}_M]$.

Proof: See [2].

This theorem suggests that reasonable estimates of the parameters $\left\{\underline{\theta}_n\right\}_{n=1}^{N}$ may be obtained by first generating an estimate $\hat{U}$ of U and then searching over all possible value of $\underline{\theta}_n$ for vectors $\underline{a}(\underline{\theta}_n)$ that are nearly orthogonal to $\hat{U}$. This may be written as

$$\hat{\underline{\theta}}_n = \arg\min_{\underline{\theta}_n} || \hat{U}^H \underline{a}(\underline{\theta}_n) ||^2 \qquad (6)$$

where $||\cdot||$ denotes the Euclidean norm. Since there is an extra degree of freedom, there is no loss of generality in assuming that $||\underline{a}(\underline{\theta}_n)|| = 1$. This also eliminates the trivial solution of (6). Note that (6) requires a multidimensional search over the parameters $\left\{\alpha_{mn}\right\}$ and $\left\{\tau_{mn}\right\}$, in contrast with the basic MUSIC method that assumes that all the parameters $\left\{\alpha_{mn}\right\}$ are equal to one or alternatively that they are known and stored in large calibration tables. The multidimensional search can be considerably simplified by decomposing $\underline{a}(\underline{\theta}_n)$ as follows:

$$\underline{a}(\underline{\theta}_n) = \Gamma(\underline{\tau}_n)\cdot\underline{\alpha}_n$$

where

$$\underline{\alpha}_n = (\alpha_{1n}, \alpha_{2n}, \cdots, \alpha_{Mn})^T ,$$

$$\Gamma(\underline{\tau}_n) = \mathrm{diag}(e^{-j\omega_1\tau_{1n}}, e^{-j\omega_1\tau_{2n}}, \cdots, e^{-j\omega_1\tau_{Mn}}) ,$$

and

$$\underline{\tau}_n = (\tau_{1n}, \tau_{2n}, \cdots, \tau_{Mn})^T .$$

Using this notation, (6) becomes

$$\hat{\underline{\theta}}_n = \arg\min_{\underline{\alpha}_n, \underline{\tau}_n} \underline{\alpha}_n^T \Gamma^H(\underline{\tau}_n) \hat{U}\hat{U}^H \Gamma(\underline{\tau}_n) \underline{\alpha}_n , \qquad (7)$$

and hence

$$\hat{\underline{\tau}}_n = \arg\min_{\underline{\tau}_n} \delta^{min}\{C(\underline{\tau}_n)\} , \tag{8.a}$$

$$\hat{\underline{\alpha}}_n = \underline{W}^{min} , \tag{8.b}$$

where $\delta^{min}\{C(\underline{\tau}_n)\}$ is the smallest eigenvalue of the matrix $C(\underline{\tau}_n)$ given by

$$C(\underline{\tau}_n) = \mathrm{Re}\left\{\Gamma^H(\underline{\tau}_n)\hat{U}\hat{U}^H\Gamma(\underline{\tau}_n)\right\} , \tag{9}$$

and $\underline{W}^{min}$ is the associated normalized eigenvector. Equation (8) requires a search over the space of vectors $\underline{\tau}_n$, induced by all possible individual source locations. In the basic MUSIC method equation (8) is simply

$$\hat{\underline{\tau}}_n = \arg\min_{\underline{\tau}_n} \left|\left|\hat{U}^H\underline{a}(\underline{\tau}_n)\right|\right|^2$$

where the vector $\underline{a}(\underline{\tau}_n)$ is only a function of the delays $\underline{\tau}_n$ and not of the intensities. That means that for every possible $\underline{\tau}_n$ the intensities are assumed to be equal to one, or a corresponding vector $\underline{\alpha}_n$ is known (stored in memory) and is used to construct $\underline{a}(\underline{\tau}_n)$.

The proposed algorithm may be summarized as follows:

(a) Estimate the observation covariance matrix:

$$\hat{R}_x = \frac{1}{J}\sum_{j=1}^{J} \underline{X}(j)\underline{X}(j)^H .$$

(b) Find the M-N eigenvectors, $\left\{\hat{\underline{u}}_i\right\}$, corresponding to the smallest M-N eigenvalues of the pencil $(\hat{R}_x,\Sigma_0)$, and construct the matrix:

$$\hat{U} = \left[\hat{\underline{u}}_{N+1},\hat{\underline{u}}_{N+2},\cdots,\hat{\underline{u}}_M\right] .$$

(c) Evaluate, for all possible source locations, the "spatial spectrum" given by:

$$P(\underline{\tau}) = \frac{1}{\delta^{min}\{C(\underline{\tau})\}}$$

where $C(\underline{\tau})$ is defined by (9).

(d) Select the N highest peaks of $P(\underline{\tau})$. The corresponding values of $\underline{\tau}$ describe the source locations, and the corresponding eigenvectors describe the intensity vectors $\left\{\underline{\alpha}_n\right\}$.

This conceptually simple algorithm requires, in step (c), considerably more computational effort than the basic MUSIC method. However, the results, illustrated in the next section, justify this effort. Methods for reducing the computational load are still under investigation.

III Examples

To illustrate the behavior of the algorithms, let us consider two examples:

<u>Example 1</u>

Consider a uniform linear array of five sensors separated by half a wavelength of the actual narrowband source signals. The sources are two narrowband emitters located in the far-field of the array. In this case, if γ_n denotes the bearing of the n-th source, n = 1,2, relative to the perpendicular to the array baseline, the differential delay is given by $\tau_{mn} = (m-1)\pi\sin(\gamma_n)$. The first source at a bearing of -9 degrees was observed with the intensity vector $\underline{\alpha}_1^T = [1,1,1,1,1]$; the second source at a bearing of 11 degrees was observed with $\underline{\alpha}_2^T = [1,.8,.6,.4,.2]$. In this case the difference in intensity may be viewed as caused by the directional patterns of the sensors. We generated 100 independent samples at SNR of 20 db.

The spatial spectrum, $P(\gamma)$, is plotted versus the angle of arrival (bearing) in Figure 1. Two sharp peaks are observed at -9 degrees and 11 degrees. The associated estimates of the intensity vectors are very close to the right result. The spurious low peak at 3 degrees is associated with a vector $\hat{\underline{\alpha}}$ containing non-physical negative components and therefore can be immediately eliminated. For comparison, we plotted the result of the basic MUSIC algorithm [1] in Figure 2. Note that only one of the sources conforms with the assumptions of MUSIC, and in this case only one peak is observed in the MUSIC output.

Example 2

Consider Example 1 except that now $\underline{\alpha}_1{}^T = \underline{\alpha}_2{}^T = (1,1,1,1,1)$, $\gamma_1 = 11^\circ$, $\gamma_2 = 25^\circ$ and SNR = 50 db. In this case the classical MUSIC algorithm works well, as the intensities match the MUSIC assumptions. The spatial spectrum of the proposed method is plotted in Figure 3. We observe two peaks at 11° and 25° and three more spurious peaks. The two peaks on the left side are associated with non-physical intensity vectors (containing negative components) and therefore can be eliminated by post processing. The spurious peak at 18° is associated with an acceptable $\hat{\underline{\alpha}}$ and therefore is an ambiguous solution.

Ambiguous solutions occur whenever the surface spanned by $\underline{a}(\theta)$ ("array manifold") intersects, or is very close to, the signal subspace (the space spanned by the columns of A) in more than N points [1]. In the above example one can predict the ambiguous solution if it is known that $\underline{\alpha}_1 \cong \underline{\alpha}_2$. Therefore an intelligent post processor may eliminate the ambiguous solution using the estimates $\hat{\underline{\alpha}}_1 \cong \hat{\underline{\alpha}}_2$ and the low probability that $\underline{\alpha}^T = (0.77, 0.95, 1.0, 0.92, 0.72)$.

IV Conclusions

In this communication the eigenstructure approach has been used to obtain estimates of source locations as well as estimates of the intensity vectors $\left\{\underline{\alpha}_n\right\}$, simultaneously. We have shown that the basic MUSIC method does not perform well when the vectors $\left\{\underline{\alpha}_n\right\}$ are not known a priori. The estimates of $\left\{\underline{\alpha}_n\right\}$ may be useful in their own right, but their estimation is essential, even if one is only interested in the source locations, in cases where it is not appropriate to assume omnidirectionality. For example, whenever a source is in the near field of the array, its radiation pattern can rarely be assumed omnidirectional. This is also important in applications in which it is unrealistic to assume that the radiation pattern of each sensor is accurately known (this usually requires frequent re-calibration and a large memory).

We observed that in some cases post-processing is required to eliminate spurious solutions and also ambiguous solutions. The elimination of spurious solutions which are associated with non-physical intensity vectors is relatively easy. On the other hand elimination of ambiguous solutions that have acceptable intensity vectors is much more complicated, and it must be based on a close examination of all the results and on a comparison with any available prior knowledge. The appearance of ambiguous peaks in certain cases and their elimination is still an open subject of research.

References

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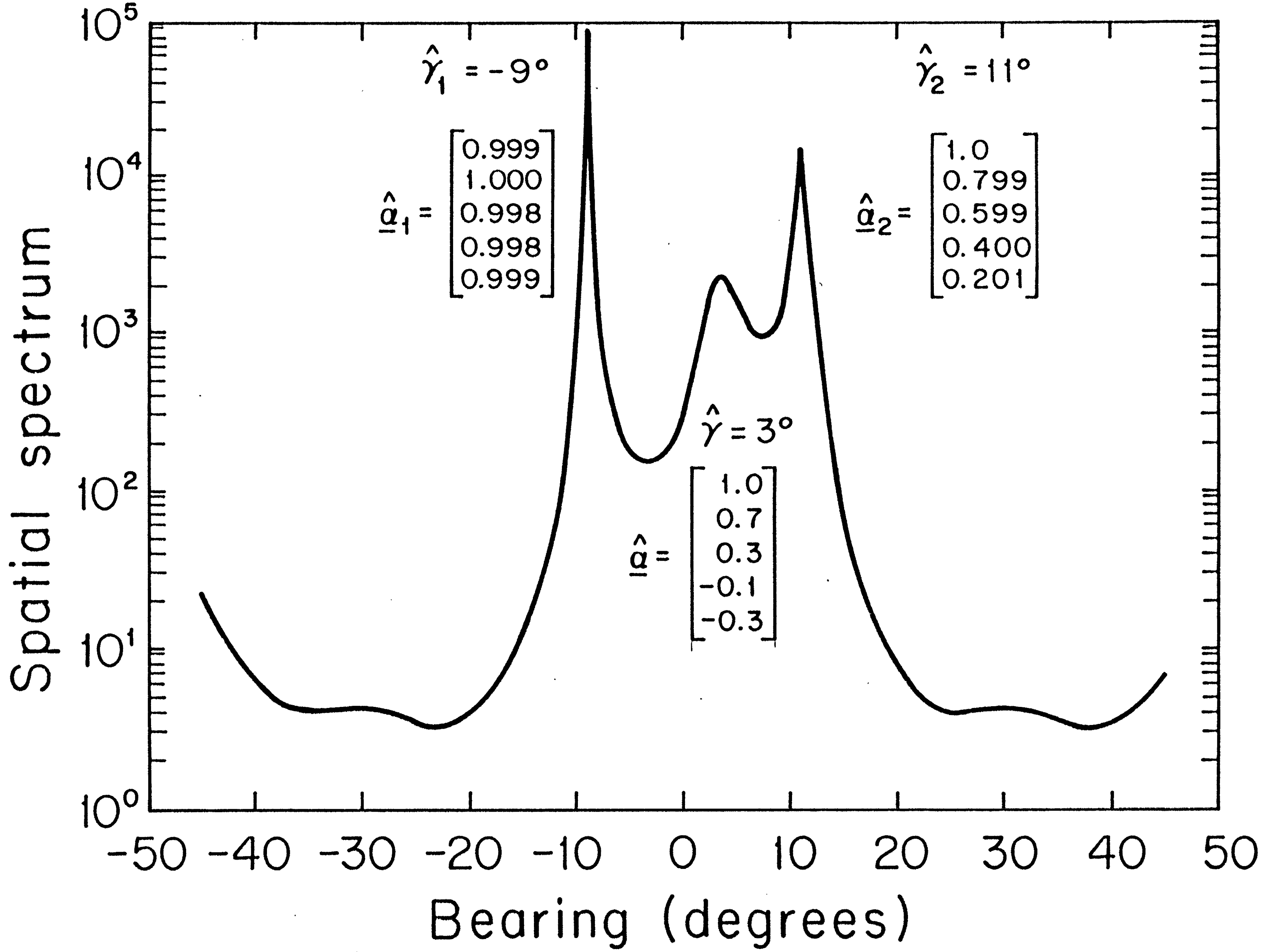
[4] M. Wax, T.J. Shan, T. Kailath, "Spatio-Temporal Spectral Analysis by Eigenstructure Methods," IEEE Trans. on Acoustics, Speech and Signal-Processing, Vol. ASSP-32, No. 4, pp. 817-827, August 1984.

Figure Captions

Figure 1: Spatial spectrum of the proposed procedure for two far-field sources.

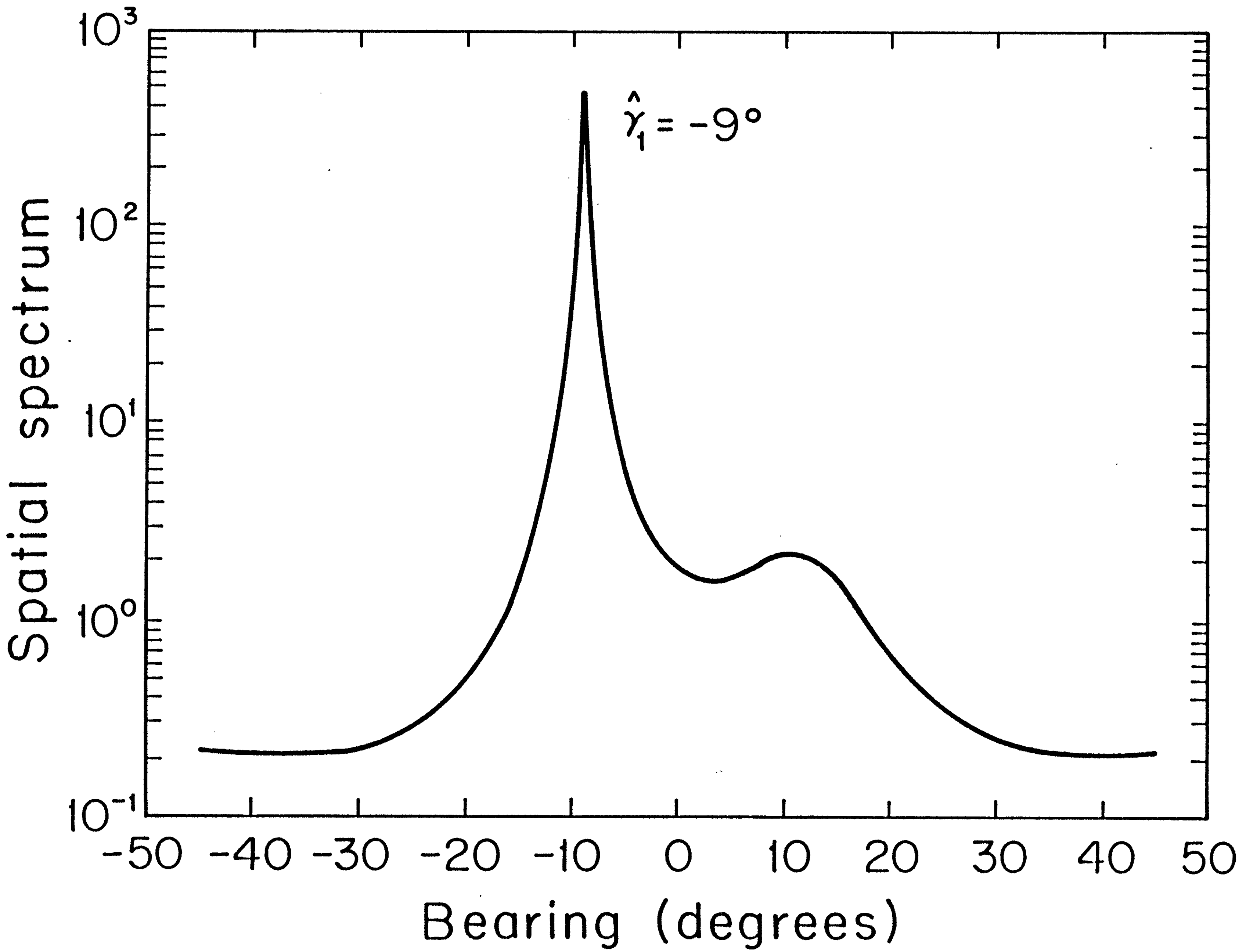
Figure 2: Spatial spectrum of the MUSIC procedure for the case of Figure 1.

Figure 3: Spatial spectrum of the proposed procedure for two far-field sources with equal intensity vectors.



Spatial spectrum
10^0
10^1
10^2
10^3
10^4
10^5
-50 -40 -30 -20 -10 0 10 20 30 40 50
Bearing (degrees)
$\hat{\gamma}_1 = -9°$
$\underline{\hat{\alpha}}_1 = \begin{bmatrix} 0.999 \\ 1.000 \\ 0.998 \\ 0.998 \\ 0.999 \end{bmatrix}$
$\hat{\gamma}_2 = 11°$
$\underline{\hat{\alpha}}_2 = \begin{bmatrix} 1.0 \\ 0.799 \\ 0.599 \\ 0.400 \\ 0.201 \end{bmatrix}$
$\hat{\gamma} = 3°$
$\underline{\hat{\alpha}} = \begin{bmatrix} 1.0 \\ 0.7 \\ 0.3 \\ -0.1 \\ -0.3 \end{bmatrix}$

Fig 1



Spatial spectrum
10^3
10^2
10^1
10^0
10^{-1}
$\hat{\gamma}_1 = -9°$
-50 -40 -30 -20 -10 0 10 20 30 40 50
Bearing (degrees)

Fig 2

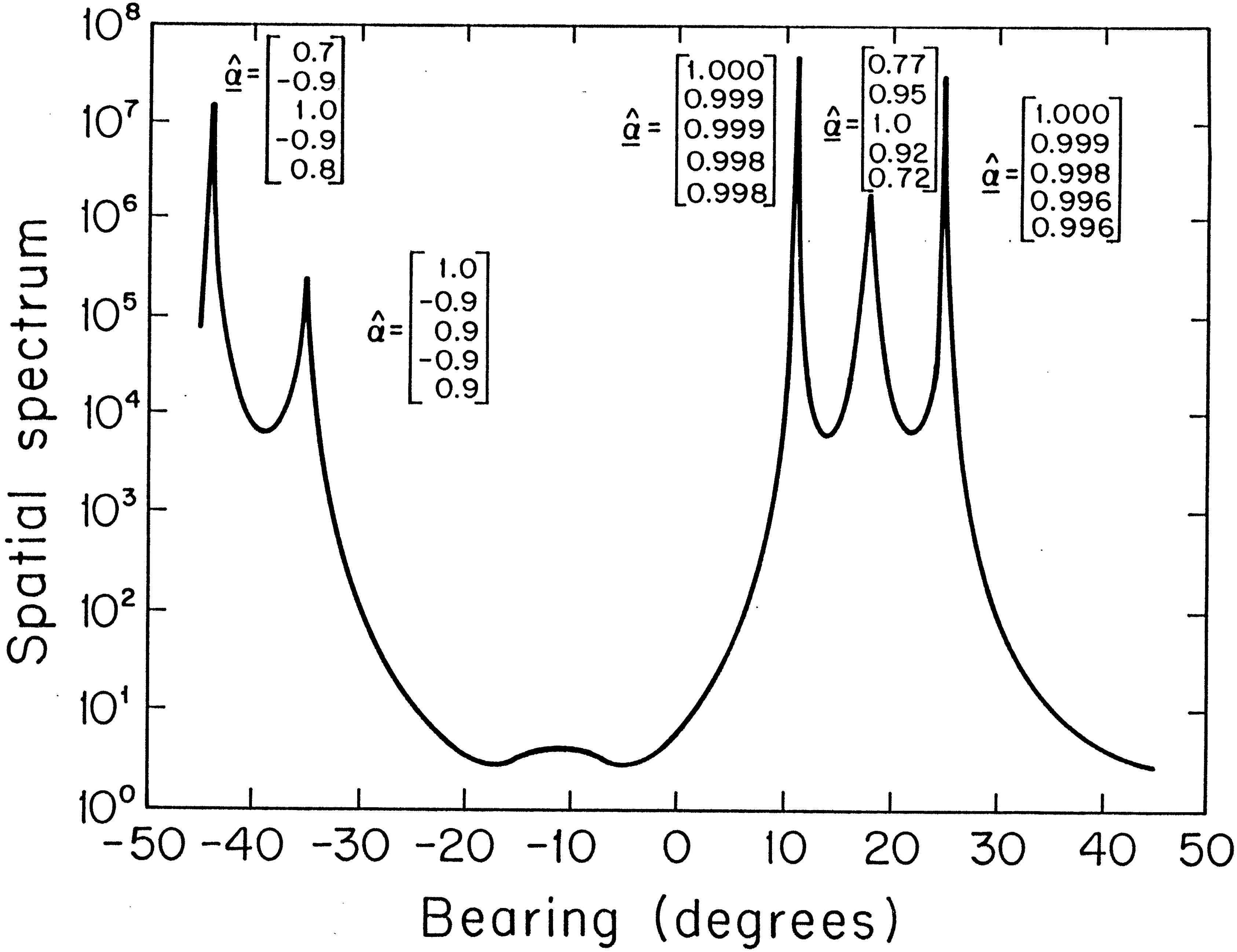

Spatial spectrum
10^8
10^7
10^6
10^5
10^4
10^3
10^2
10^1
10^0
-50 -40 -30 -20 -10 0 10 20 30 40 50
Bearing (degrees)
α̂ = [0.7, -0.9, 1.0, -0.9, 0.8]
α̂ = [1.0, -0.9, 0.9, -0.9, 0.9]
α̂ = [1.000, 0.999, 0.999, 0.998, 0.998]
α̂ = [0.77, 0.95, 1.0, 0.92, 0.72]
α̂ = [1.000, 0.999, 0.998, 0.996, 0.996]

Fig 3